On the decoding of quasi-BCH codes

Morgan Barbier*
morgan.barbier@unicaen.fr

Clément Pernet†
clement.pernet@imag.fr

Guillaume Quintin‡
quintin@lix.polytechnique.fr

2012/12/21

Abstract

In this paper we investigate the structure of quasi-BCH codes. In the first part of this paper we show that quasi-BCH codes can be derived from Reed-Solomon codes over square matrices extending the known relation about classical BCH and Reed-Solomon codes. This allows us to adapt the Welch-Berlekamp algorithm to quasi-BCH codes. In the second part of this paper we show that quasi-BCH codes can be seen as subcodes of interleaved Reed-Solomon codes over finite fields. This provides another approach for decoding quasi-BCH codes.

keywords: Quasi-cyclic code, quasi-BCH code, BCH code, Reed-Solomon, interleaved code

1 Introduction

Many codes with best known minimum distances are quasi-cyclic codes or derived from them [LS03, Gra07]. This family of codes is therefore very interesting. Quasi-cyclic codes were studied and applied in the context of McEliece's cryptosystem [McE78, BCGO09] and Niederreiter's [Nie86, LDW94]. They permit to reduce the size of keys in opposition to Goppa codes. However, since the decoding of random quasi-cyclic codes is difficult, only quasi-cyclic alternant codes were proposed for the latter cryptosystem. The high structure of alternant codes is actually a weakness and two cryptanalysis were proposed in [FOPT10, UL10]

*Université de Caen - GREYC, Boulevard Maréchal Juin, BP 5186, 14032 Caen
†Univ. Joseph Fourier INRIA/LIG-MOAIS, 51 avenue Jean Kuntzmann, 38330 Montbonnot
‡École polytechnique - LIX, 91128 Palaiseau Cedex

1.1 Our contributions

In this paper we investigate the structure of quasi-BCH codes. In the first part of this paper we show that quasi-BCH codes can be derived from Reed-Solomon codes over square matrices. It is well known that BCH codes can be obtained from Reed-Solomon codes [MS86, Theorem 2, page 300]. We extend this property to quasi-BCH codes which allows us to adapt the Welch-Berlekamp algorithm to quasi-BCH codes.

Theorem 1. *Let $\Gamma \in M_{\ell\times\ell}(\mathbb{F}_{q^s})$ be a primitive m-th root of unity and $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$. Then there exists a RRS code $\mathcal{R}$ over the ring $M_{\ell\times\ell}(\mathbb{F}_{q^s})$ with parameters $[n, n-\delta+1]_{M_{\ell\times\ell}(\mathbb{F}_{q^s})}$ and a $\mathbb{F}_q$-linear, F_q-isometric embedding $\psi : \mathcal{C} \to \mathcal{R}$.*

In the second part we show that quasi-BCH codes can be seen as subcodes of interleaved Reed-Solomon codes.

Theorem 2. *The quasi-BCH code $\mathcal{C}$ over $\mathbb{F}_q$ is an interleaved code of ℓ subcodes of Reed-Solomon codes over $\mathbb{F}_{q^{s'}}$ in the following sense: there exists ℓ Reed-Solomon codes $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ over $\mathbb{F}_q$ and an isometric isomorphism from $\mathcal{C}$, equipped with the ℓ-block distance, to a subcode of the interleaved code with respect to $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$.*

1.2 Related work

In [LF01, LS01], ℓ-quasi-cyclic codes of length $m\ell$ are seen as R-submodules of R^ℓ for a certain ring R. However, in [LF01], Gröbner bases are used in order to describe polynomial generators of quasi-cyclic codes whereas in [LS01], the authors decompose quasi-cyclic codes as direct sums of shorter linear codes over various extensions of $\mathbb{F}_q$ (when $\gcd(m, q) = 1$). This last work leads to an interesting trace representation of quasi-cyclic codes. In [CCN10], the approach is more analogous to the cyclic case. The authors consider the factorization of $X^m - 1 \in M_\ell(F_q)[X]$ with reversible polynomials in order to construct ℓ-quasi-cyclic codes canceled by those polynomials and called $\Omega(P)$-codes. This leads to the construction of self-dual codes and codes beating known bounds. But the factorization of univariate polynomials over a matrix ring remains difficult. In [Cha11] the author gives an improved method for particular cases of the latter factorization problem.

2 Prerequisites

2.1 Reed-Solomon codes over rings

We recall some basic definitions of Reed-Solomon codes over rings in this section. We let A be a ring with identity, we denote by $A^\times$ the *group of units* of A and by $Z(A)$ the *center* of A, the commutative subring of A consisting of all the elements of A which commutes with all the other elements of A. We denote by

$A[X]$ the ring of polynomials over A and by $A[X]_{<k}$ the polynomials over A of degree at most $k-1$.

Definition 1. *Let*

$$f = \sum_{i=0}^{d} f_i X^i \in A[X]$$

be a polynomial with coefficients in A and $a \in A$. We call left evaluation of f at a *the quantity*

$$f(a) := \sum_{i=0}^{d} f_i a^i \in A$$

and right evaluation of f at a *the quantity*

$$(a)f := \sum_{i=0}^{d} a^i f_i \in A.$$

Remark 1. *For $f, g \in A[X]$ and $a \in A$, we obviously have $f(a) = (a)f$ whenever $a \in Z(A)$, $(f+g)(a) = f(a) + g(a)$, $(a)(f+g) = (a)f + (a)g$. If a commutes with all the coefficients of g we also have $(fg)(a) = f(a)g(a)$ and $(a)(gf) = (a)g(a)f$.*

Definition 2. *Let $0 < k \leq n$ be two integers. Let $(x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$ be two vectors of A^n be such that $x_i - x_j \in A^\times$ and $x_i x_j = x_j x_i$ for all $i \neq j$ and $v_i \in A^\times$ for all i.*

The left *submodule of A^n generated by the vectors*

$$(f(x_1) \cdot v_1, \ldots, f(x_n) \cdot v_n) \in A^n \text{ with } f \in A[X]_{<k}$$

is called a left generalized Reed-Solomon code (LGRS) over A with parameters $[v, x, k]_A$ *or $[n, k]$ if there is no confusion on x and v.*

The right *submodule of A^n generated by the vectors*

$$(v_1 \cdot (x_1)f, \ldots, v_n \cdot (x_n)f) \in A^n \text{ with } f \in A[X]_{<k}$$

is called a right generalized Reed-Solomon code (RGRS) over A with parameters $[v, x, k]_A$ *or $[n, k]$ if there is no confusion on x and v. The vector x is called the* support *of the code. If $v = (1, \ldots, 1)$, the codes constructed above are called left Reed-Solomon (LRS) and right Reed-Solomon (RRS) codes.*

Definition 3. *Let $x = (x_1, \ldots, x_n) \in A^n$. We call the* Hamming weight of x *the number of nonzero coordinates.*

$$w(x) := w(x_1, \ldots, x_n) = |\{i : x_i \neq 0\}|.$$

Let $y = (y_1, \ldots, y_n) \in A^n$. The Hamming distance between x and y *is*

$$d(x, y) = w(x - y) = |\{i : x_i \neq x_j\}|.$$

The minimum distance *of any subset $S \subseteq A^n$ is defined as*

$$\min \{d(x, y) : x, y \in S \text{ and } x \neq y\}.$$

Proposition 1. *A LGRS (resp. RGRS) code is a free left (resp. right) submodule of A^n. A LGRS (resp. RGRS) code with parameters $[n, k]$ has minimum distance $n - k + 1$.*

Proof. It suffices to see that the maps

$$\begin{array}{rcl} A^n & \longrightarrow & A^n \\ (a_1, \ldots, a_n) & \longmapsto & (a_1 v_1, \ldots, a_n v_n) \\ (a_1, \ldots, a_n) & \longmapsto & (v_1 a_1, \ldots, v_n a_n) \end{array}$$

are respectively left and right isometric automorphisms of A^n. □

2.2 Quasi cyclic and quasi BCH codes

Quasi cyclic codes form an important family of codes defined as follow.

Definition 4. *Let $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ to be the left cyclic shift defined by*

$$T(c_1, c_2, \ldots, c_n) = (c_2, c_3, \ldots, c_1).$$

*We call ℓ-*quasi-cyclic code over $\mathbb{F}_q$ of length n *any code of length n over $\mathbb{F}_q$ stable by T^ℓ. If the context is clear we will simply say ℓ-*quasi-cyclic code.

We will focus in this paper on quasi-BCH codes which form a subfamily of quasi-cyclic codes. They can be seen as a generalization of BCH codes in the context of quasi-cyclic codes. For we need primitive roots of unity defined in a extension of $\mathbb{F}_q$, say $\mathbb{F}_{q^s}$ to construct BCH codes over $\mathbb{F}_q$.

Proposition 2. *Then there exists a primitive $q^{s\ell}-1$-th root of unity in $M_\ell(\mathbb{F}_{q^s})$.*

Proof. The proof can be found in [BCQ12b, Proposition 16, page 911]. □

Definition 5. *Let Γ be a primitive m-th root of unity in $M_\ell(\mathbb{F}_{q^s})$ and $\delta \leq m$. We define the ℓ-quasi-BCH code of length $m\ell$, with respect to Γ, with designed minimum distance δ, over $\mathbb{F}_q$ by*

$$\text{Q-BCH}_q(m, \ell, \delta, \Gamma) := \left\{ (c_1, \ldots, c_m) \in (\mathbb{F}_q^\ell)^m : \sum_{j=0}^{m-1} (\Gamma^i)^j (c_{j+1})^T = 0 \text{ for } i = 1, \ldots, \delta - 1 \right\}.$$

Note that $\text{Q-BCH}_q(m, \ell, \delta, \Gamma)$ *is a quasi-cyclic code.*

Definition 6. *The* ℓ-block weight *of $(x_{11}, \ldots, x_{1\ell}, \ldots, x_{m1}, \ldots, x_{m\ell}) \in \mathbb{F}_q^{m\ell}$ is defined to be*

$$\text{Block-w}_\ell(x) := |\{i : (x_{i1}, \ldots, x_{i\ell}) \neq 0\}|.$$

The ℓ-block distance *between $x, y \in \mathbb{F}_q^{m\ell}$ is defined to be* $\text{Block-w}_\ell(x - y)$.

3 Reed-Solomon codes and quasi-BCH codes

3.1 The relation between quasi-BCH and Reed-Solomon codes

We show in this section that under certain assumptions on the support of Reed-Solomon codes, the dual of a LRS code is a RRS code. From this fact we show that quasi-BCH can be constructed from Reed-Solomon codes over square matrices rings. In this Subsection we let A designate a finite ring with identity.

Definition 7. *Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors of A^n. The* inner product *is defined as*

$$\langle x, y \rangle := \sum_{i=0}^{n} x_i y_i.$$

Remark 2. *Let S be a subset of A^n. Then the set $\{x \in A^n : \forall s \in S, \langle s, x \rangle = 0\}$ denoted by $S^\perp$ is called the* right dual *of S and is a right submodule of A^n. Similarly, Let S be a subset of A^n. Then the set $\{x \in A^n : \forall s \in S, \langle x, s \rangle = 0\}$ denoted by ${}^\perp S$ is called the* left dual *of S and is a left submodule of A^n. Note that for all $x, y \in A^n$ and $\mu \in A$ we have $\mu \langle x, y \rangle = \langle \mu x, y \rangle$ and $\langle x, y \rangle \mu = \langle x, y\mu \rangle$.*

Definition 8. *We say that $a \in A$ is a* primitive m-th root of unity *if $a^m = 1$ and $\forall 0 \leq i < m, (a^i - 1) \in A^\times$.*

Remark 3. *Let $x = (1, \gamma, \gamma^2, \dots, \gamma^{m-1}) \in A^m$ where γ is a primitive m-th root of unity. Then a RRS or LRS code whose support is x is cyclic.*

Proposition 3. *Let $\gamma \in A$ be a primitive m-th root of unity. Let $x = (1, \gamma, \gamma^2, \dots, \gamma^{m-1}) \in A^n$. Then the right (resp. left) dual of the LGRS (resp. RGRS) code with parameters $[x, x, k]_A$ is the RRS (resp. LRS) code with parameters $[x, n-k]_A$.*

Proof. We denote respectively by $\mathcal{L}$ and $\mathcal{R}$ the left generalized Reed-Solomon code with parameters $[x, x, k]_A$ and the right Reed-Solomon code with parameters $[x, n-k]_A$.

First note that $\mathcal{L}$ is generated by the vectors

$$(1, \gamma^i, \gamma^{2i}, \dots, \gamma^{(m-1)i}) \text{ for } i = 1, \dots, k$$

and that $\mathcal{R}$ is generated by the vectors

$$(1, \gamma^i, \gamma^{2i}, \dots, \gamma^{(m-1)i}) \text{ for } i = 0, \dots, n-k-1.$$

And we have for $0 \leq i + j < n - 1$ in the commutative ring $Z(A)[\gamma]$

$$\sum_{i=0}^{m-1} \gamma^{(i+1)\ell} \cdot \gamma^{j\ell} = \sum_{i=0}^{m-1} \left(\gamma^{i+j+1}\right)^\ell = \frac{1 - \left(\gamma^{i+j+1}\right)^m}{1 - \gamma^{i+j+1}} = 0.$$

Therefore, by Proposition 1 and Remark 2, $\mathcal{L}^\perp \subseteq \mathcal{R}$ and ${}^\perp\mathcal{R} \subseteq \mathcal{L}$.

Again by Proposition 1 and Remark 2 an element $x \in A^n$ lies in $\mathcal{L}^\perp$ if and only if

$$\left[\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \gamma & \gamma^2 & \dots & \gamma^{m-1} \\ 1 & \vdots & \vdots & & \vdots \\ 1 & \gamma^{k-1} & \gamma^{2(k-1)} & \dots & \gamma^{(k-1)(m-1)} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma^{m-1} \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0. \tag{1}$$

But in the commutative ring $Z(A)[\gamma]$ the matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \gamma & \gamma^2 & \dots & \gamma^{2(k-1)} \\ 1 & \vdots & \vdots & & \vdots \\ 1 & \gamma^{k-1} & \gamma^{2(k-1)} & \dots & \gamma^{(k-1)(k-1)} \end{pmatrix} \in M_{k\times k}\left(Z(A)[\gamma]\right)$$

is invertible. Therefore H is also invertible in $M_{k\times k}(A)$ and thus induces a group automorphism of A^k. If we let $x_H = (x_1, \dots, x_k)$, $x_U = (x_{k+1}, \dots, x_n)$, we can rewrite equation (1) as

$$\left(H \mid U \right) \left(\frac{x_H}{x_U}\right) = 0 \text{ and } \left(H \mid 0 \right) \left(\frac{x_H}{0}\right) = -\left(0 \mid U \right) \left(\frac{0}{x_U}\right).$$

For each choice of x_U we have only one possible value for x_H. Thus $|\mathcal{L}^\perp| = |A|^{n-k} = |\mathcal{R}|$ by Proposition 1 and therefore $\mathcal{L}^\perp = \mathcal{R}$. Similarly, we have ${}^\perp\mathcal{R} = \mathcal{L}$. □

Theorem 3. *Let $\Gamma \in M_{\ell\times\ell}(\mathbb{F}_{q^s})$ be a primitive m-th root of unity and $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$. Then there exists a RRS code $\mathcal{R}$ over the ring $M_{\ell\times\ell}(\mathbb{F}_{q^s})$ with parameters $[n, n-\delta+1]_{M_{\ell\times\ell}(\mathbb{F}_{q^s})}$ and a $\mathbb{F}_q$-linear, F_q-isometric embedding $\psi : \mathcal{C} \to \mathcal{R}$.*

Proof. A parity-check matrix of $\mathcal{C}$ is

$$H = \begin{pmatrix} I_\ell & \Gamma & \cdots & \Gamma^{m-1} \\ I_\ell & \Gamma^2 & \cdots & \Gamma^{2(m-1)} \\ \vdots & \vdots & & \vdots \\ I_\ell & \Gamma^{\delta-1} & \cdots & \Gamma^{(\delta-1)(m-1)} \end{pmatrix} \in M_{(\delta-1)\ell, m\ell}(\mathbb{F}_{q^s}).$$

Remark that H is a generator matrix of the LGRS code with parameters $[x, x, \delta-1]_{M_{\ell\times\ell}(\mathbb{F}_{q^s})}$ over the ring $M_{\ell\times\ell}(\mathbb{F}_{q^s})$ and by Proposition 3 its dual is the RRS with parameters $[x, \delta-1]_{M_{\ell\times\ell}(\mathbb{F}_{q^s})}$.

Now let

$$\begin{aligned} \psi : \mathcal{C} &\longrightarrow \left(M_{\ell\times\ell}(\mathbb{F}_{q^s})\right)^m \\ (c_{11}, \dots, c_{1\ell}, \dots, c_{m1}, \dots, c_{m\ell}) &\longmapsto \left[\begin{pmatrix} c_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{1\ell} & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} c_{m1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{m\ell} & 0 & \dots & 0 \end{pmatrix}\right]. \end{aligned}$$

Obviously, ψ is $\mathbb{F}_q$-linear, injective and isometric and by the above remark we have $\psi(\mathcal{C}) \subseteq \mathcal{R}$. $\square$

Theorem 3 generalizes the well-known [MS86, Theorem 2, page 300] relation between BCH codes and Reed-Solomon codes. The above relation will allow us to adapt the unique decoding algorithm from [BCQ12a] to quasi-BCH codes.

3.2 The Welch-Berlekamp algorithm for quasi-BCH codes

In this Subsection we let A designate a finite ring with identity. Before giving the Welch-Berlekamp decoding algorithm, we need to define what the *evaluation* of a bivariate polynomial over A is. Let $Q = \sum Q_{i,j} X^i Y^j \in A[X,Y]$ be such a polynomial. We define the *evaluation of Q at $(a,b) \in A^2$* to be

$$(a,b)Q = \sum a^i b^j Q_{i,j} \in A.$$

Be careful of the order of a, b and $Q_{i,j}$. This choice will be explained in the proof of Lemma 1. Let $f \in A[X]$, we define *the evaluation of Q at f* to be

$$(X, f(X))Q = \sum X^j (f(X))^j Q_{i,j} \in A[X].$$

As in the univariate case, the evaluation maps defined above are not ring homomorphisms in general.

Lemma 1. *Let $g \in A[X]$, $Q \in A[X,Y]$ of degree at most 1 in Y and $a \in A$. Then*

$$(a)((X, g(X))Q) = (a, (a)g)Q.$$

Proof. We write

$$\begin{aligned} Q(X,Y) &= Q_0(X) + Q_1(X)Y \\ &= Q_0(X) + \left(\sum_i Q_{1i} X^i\right) Y. \end{aligned}$$

The proof is an easy calculation:

$$\begin{aligned} (a)((X, g(X))Q) &= (a)\left(Q_0(X) + \sum_i X^i g(X) Q_{1i}\right) \\ &= (a)Q_0 + \sum_i a^i (a)g Q_{1i} \\ &= (a, (a)g)Q \text{ by definition.} \end{aligned}$$

$\square$

We let $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$, $\tau = \left\lfloor \frac{\delta-1}{2} \right\rfloor$, $n = m$, $k = n - \delta + 1$ and

$$\begin{array}{rcl} \mathrm{pr} : (M_{\ell\times\ell}(\mathbb{F}_{q^s}))^m & \longrightarrow & \mathbb{F}_q^{m\ell} \\ \left[\begin{pmatrix} a_{11}^1 & \dots & a_{1\ell}^1 \\ \vdots & & \vdots \\ a_{\ell 1}^1 & \dots & a_{\ell\ell}^1 \end{pmatrix}, \dots, \begin{pmatrix} a_{11}^m & \dots & a_{1\ell}^m \\ \vdots & & \vdots \\ a_{\ell 1}^m & \dots & a_{\ell\ell}^m \end{pmatrix}\right] & \longmapsto & (a_{11}^1, \dots, a_{\ell 1}^1, \dots, a_{11}^m, \dots, a_{\ell 1}^m). \end{array}$$

Algorithm 1 Welch-Berlekamp for quasi-BCH codes

Input: a received vector $y \in \mathbb{F}_q^{m\ell}$ with at most τ errors.
Output: the unique codeword within distance τ of y.

1: $(Z_1, \dots, Z_m) \leftarrow \psi(y)$ where ψ is the map from Theorem 3.
2: Find $Q = Q_0(X) + Q_1(X)Y \in (M_{\ell\times\ell}(\mathbb{F}_{q^s})[X])[Y]$ of degree 1 such that

1. $(\Gamma^{i-1}, Z_i)Q = 0$ for all $i = 1, \dots, m-1$,
2. $\deg Q_0 \leq n - \tau - 1$,
3. $\deg Q_1 \leq n - \tau - 1 - (k-1)$.

3: $f \leftarrow$ the unique root of Q in $(M_{\ell\times\ell}(\mathbb{F}_{q^s}))[X]_{<k}$ such that $d\big((Z_1, \dots, Z_m), ((I_\ell)f, \dots, (\Gamma^{m-1})f)\big) \leq \tau$.
4: **return** $\mathrm{pr}\big((I_\ell)f, (\Gamma)f, \dots, (\Gamma^{m-1})f\big)$.

Lemma 2. *Let $y \in \mathbb{F}_q^{m\ell}$ be a received word containing at most τ errors. Then there exists a nonzero bivariate polynomial $Q = Q_0 + Q_1 Y \in (M_{\ell\times\ell}(\mathbb{F}_{q^s}))[X, Y]$ satisfying*

1. $(\Gamma^{i-1}, Z_i)Q = 0$ *for* $i = 1, \dots, n$.
2. $\deg Q_0 \leq n - \tau - 1$.
3. $\deg Q_1 \leq n - \tau - 1 - (k-1)$.

Proof. We solve the problem with linear algebra over $\mathbb{F}_{q^s}$. We have, for each column of the solution, $n\ell$ equations and $\ell\,[(n-\tau) + (n - \tau - (k-1))] = \ell(n+1)$ unknowns by Proposition 1. □

Lemma 3. *Let $Q \in (M_{\ell\times\ell}(\mathbb{F}_{q^s}))[X, Y]$ satisfying the three conditions of Lemma 2 and $f \in (M_{\ell\times\ell}(\mathbb{F}_{q^s}))[X]_{<k}$ be such that $d((Z_1, \dots, Z_m), ((I_\ell)f, \dots, (\Gamma^{m-1})f)) \leq \tau$. Then $(X, f(X))Q = 0$.*

Proof. The polynomial $(X, f(X))Q$ has degree at most $n - \tau - 1$. By Lemma 1 we have $(\Gamma^{i-1})((X, f(X))Q) = (\Gamma^{i-1}, (\Gamma^{i-1})f)Q = (\Gamma^{i-1}, Z_i)Q = 0$ for at least $n - \tau$ values of $i \in \{1, \dots, n\}$. And therefore we must have $(X, f(X))Q = 0$. □

Proposition 4. *Algorithm 1 works correctly as expected and can correct up to $\left\lfloor \frac{\delta-1}{2} \right\rfloor$ errors.*

Proof. This is a direct consequence of Lemmas 2 and 3. □

4 Quasi-BCH codes as interleaved codes

In this Section we prove that quasi BCH codes can be viewed as an interleaving of classical BCH codes. We fix for this Section $\Gamma \in M_{\ell\times\ell}(\mathbb{F}_{q^s})$ a primitive m-th root of unity and $\mathcal{C} = \text{Q-BCH}_q(m, \ell, \delta, \Gamma)$. We first recall the definition of interleaved codes.

Definition 9. *Let $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ be error correcting codes over $\mathbb{F}_q$. The* interleaved code $\mathcal{C}$ with respect to *$\mathcal{C}_1, \dots, \mathcal{C}_\ell$ is a subset of $M_{\ell\times m}(\mathbb{F}_q)$, equipped with the ℓ-bloc distance with respect to the columns, such that $c \in \mathcal{C}$ if and only if the i-th row of c is a codeword of $\mathcal{C}_i$ for $i = 1, \dots, \ell$.*

Lemma 4. *The matrix Γ diagonalizes over an extension of $\mathbb{F}_{q^s}$ and its eigenvalues are all primitive m-th roots of unity.*

Proof. Let $\mathbb{F}_{q^{s'}} \supseteq \mathbb{F}_{q^s}$ be the splitting field of $X^m - 1$. The polynomial $X^m - 1$ is a multiple of the minimal polynomial $\mu(X)$ of Γ. Hence the egeinvalues of Γ are m-roots of unity. Let $P \in \text{GL}_\ell(\mathbb{F}_{q^{s'}})$ be such that $P^{-1}\Gamma P$ is diagonal. Now if an eigenvalue λ_i of Γ has order $d < m$, then

$$P^{-1}(\Gamma^d - I_\ell)P = \begin{pmatrix} \lambda_1^d & & & & \\ & \ddots & & & \\ & & \lambda_i^d & & \\ & & & \ddots & \\ & & & & \lambda_\ell^d \end{pmatrix} - I_\ell$$

is singular as its i-th diagonal element would be zero. Consequently $\Gamma^d - I_\ell \notin \text{GL}_\ell(\mathbb{F}_{q^{s'}})$ which is absurd. □

Theorem 4. *The quasi-BCH code $\mathcal{C}$ over $\mathbb{F}_q$ is an interleaved code of ℓ subcodes of Reed-Solomon codes over $\mathbb{F}_{q^{s'}}$ in the following sense: there exists ℓ Reed-Solomon codes $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ over $\mathbb{F}_q$ and an isometric isomorphism from $\mathcal{C}$, equipped with the ℓ-block distance, to a subcode of the interleaved code with respect to $\mathcal{C}_1, \dots, \mathcal{C}_\ell$.*

Proof. We take the notation of the proof of Lemma 4. Recall that

$$H = \begin{pmatrix} I_\ell & \Gamma & \cdots & \Gamma^{m-1} \\ I_\ell & \Gamma^2 & \cdots & \Gamma^{2(m-1)} \\ \vdots & \vdots & & \vdots \\ I_\ell & \Gamma^{\delta-1} & \cdots & \Gamma^{(\delta-1)(m-1)} \end{pmatrix} \in M_{(\delta-1)\ell, m\ell}(\mathbb{F}_{q^s})$$

is a parity check matrix for $\mathcal{C}$ (proof of Theorem 3). By Lemma 4 we have that

$$(c_{11},\dots,c_{1\ell},\dots,c_{m1},\dots,c_{m\ell})\in\mathcal{C}\iff$$
$$\begin{pmatrix}P^{-1}&&\\&\ddots&\\&&P^{-1}\end{pmatrix}\begin{pmatrix}I_\ell&\Gamma&\cdots&\Gamma^{m-1}\\I_\ell&\Gamma^2&\cdots&\Gamma^{2(m-1)}\\\vdots&\vdots&&\vdots\\I_\ell&\Gamma^{\delta-1}&\cdots&\Gamma^{(\delta-1)(m-1)}\end{pmatrix}\begin{pmatrix}P&&\\&\ddots&\\&&P\end{pmatrix}\times$$
$$\left[\begin{pmatrix}P^{-1}&&\\&\ddots&\\&&P^{-1}\end{pmatrix}\begin{pmatrix}c_{11}\\\vdots\\c_{1\ell}\\\vdots\\c_{m1}\\\vdots\\c_{m\ell}\end{pmatrix}\right]=0$$
$$\text{and }(c_{11},\dots,c_{1\ell},\dots,c_{m1},\dots,c_{m\ell})\in\mathbb{F}_q^{m\ell}$$

Let

$$\begin{pmatrix}v_{11}\\\vdots\\v_{1\ell}\\\vdots\\v_{m1}\\\vdots\\v_{m\ell}\end{pmatrix}=\begin{pmatrix}P^{-1}&&\\&\ddots&\\&&P^{-1}\end{pmatrix}\begin{pmatrix}c_{11}\\\vdots\\c_{1\ell}\\\vdots\\c_{m1}\\\vdots\\c_{m\ell}\end{pmatrix}\tag{2}$$

Denote by σ the application defined by equation (2). Then

$$(c_{11},\dots,c_{1\ell},\dots,c_{m1},\dots,c_{m\ell})\in\mathcal{C}\iff$$
$$\sigma^{-1}(v_{11},\dots,v_{1\ell},\dots,v_{m1},\dots,v_{m\ell})\in\mathbb{F}_q^{m\ell}\text{ and for }i=1,\dots,\ell$$
$$\begin{pmatrix}1&\lambda_i&\dots&\lambda_i^{m-1}\\1&\lambda_i^2&\dots&\lambda_i^{2(m-1)}\\\dots&\dots&\dots&\dots\\1&\lambda_i^{\delta-1}&\dots&\lambda_i^{(\delta-1)(m-1)}\end{pmatrix}\begin{pmatrix}v_{1i}\\\vdots\\v_{mi}\end{pmatrix}=0.\tag{3}$$

Then it is straightforward that σ is an isometric isomorphism from $\mathcal{C}$ equipped with the ℓ-block distance and $\sigma(\mathcal{C})$, which is by equation (3) a subcode of the interleaved code with respect to ℓ subcodes of Reed-Solomon codes over $\mathbb{F}_q$. For $i=1,\dots,\ell$ take $\mathcal{C}_i$ to be the Reed-Solomon code defined by the parity check matrix of equation (3). □

Note that if the minimal polynomial of Γ has degree one: $\Gamma = X-\lambda$, then $s'=s$ and Γ diagonalizes as λI_ℓ. Consequently the Reed-Solomon codes

$\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ are isomorphic, as they are defined by the same control equations in equation (3). In such a case, we can apply the result on the correction capacity for interleaved Reed-Solomon codes [SSB06, BKY07].

Corollary 1. *There exists a decoding algorithm that is guaranteed to correct up to* $\frac{\delta-1}{2}$ *errors. In particular, if the minimal polynomial of* Γ *has degree* 1 *over* $\mathbb{F}_{q^s}$ *then it can correct up to* $\frac{\ell}{\ell+1}(\delta-1)$ *errors with high probability.*

Proof. Taking the notation of Theorem 4 and if $y = c+e$ is a received word, one can decode $\sigma(y)$ with the decoding algorithms of $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ obtaining $c' \in \mathbb{F}_{q^{s'}}^{m\ell}$. Then $c = \sigma^{-1}(c')$.

If the minimal polynomial of Γ has degree 1, then $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_\ell$ and one can apply the algorithm of [BKY07] or [SSB06]. □

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